Symmetric vs. Sum Capacity of Rayleigh MAC

or

Probability of Achieving Fairness for Free

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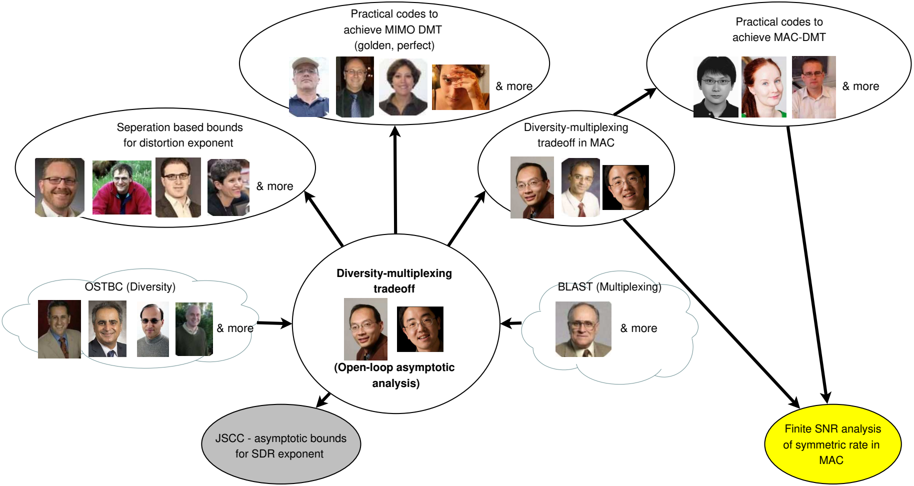
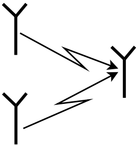
February 14th, 2018
2018 Information Theory and Applications Workshop


Practical codes to
achieve MIMO DMT
(golden, perfect)
& more
Practical codes to
achieve MAC-DMT
& more
Seperation based bounds
for distortion exponent
& more
Diversity-multiplexing
tradeoff in MAC
OSTBC (Diversity)
& more
Diversity-multiplexing
tradeoff
(Open-loop asymptotic
analysis)
BLAST (Multiplexing)
& more
JSCC - asymptotic bounds
for SDR exponent
Finite SNR analysis
of symmetric rate in
MAC

Channel Model

- MAC: $$y = \sum_{i=1}^{N} h_i x_i + z$$



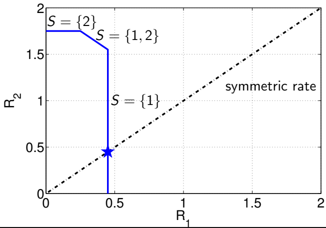
- CSI at Rx
- Equal average transmission power per antenna: $P = 1$
- $z \sim \mathcal{CN}(0,1)$
- $h_i \sim \sqrt{\mathrm{SNR}} \cdot \mathcal{CN}(0,1)$ and i.i.d. (symmetric setting)

Definitions

- Sum capacity: $C_{\text{sum}} = \log\left(1 + \sum |h_i|^2\right)$
- Capacity region (set of constrains):

$$C(\mathbf{h}) = \sum_{i\in S} R_i \leq \log\left(1 + \sum_{i\in S} |h_i|^2\right), \; S \subseteq \{1,\ldots,N\}$$

- Symmetric capacity:
  - $C_{\text{sym}} = \max_{\mathbf{R}\in C(\mathbf{h})} \min(R_1,\ldots,R_N) = \min_{S\subseteq\{1,\ldots,N\}} \frac{1}{|S|} \log\left(1 + \sum_{i\in S} |h_i|^2\right)$
  - $C_{\Sigma-\text{sym}} = N \cdot C_{\text{sym}}$



Need to analyze the bottleneck !

Symmetric vs. Sum Capacity

- $C_{\Sigma-\mathrm{sym}} = C_{\mathrm{sum}} \Rightarrow$ fairness comes for free!
- But what are the chances of that happening?
- **Q1**: Probability that $C_{\Sigma-\mathrm{sym}} = C_{\mathrm{sum}} = \log\left(1 + \sum_{i=1}^{N} |h_i|^2\right)$ is?
- We analyze the probability given C_{sum}
- Let's start with a concrete example: $C_{\mathrm{sum}} = 2$

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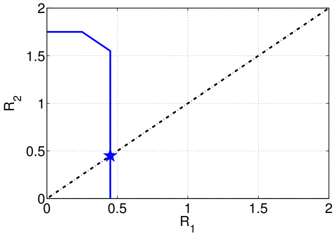

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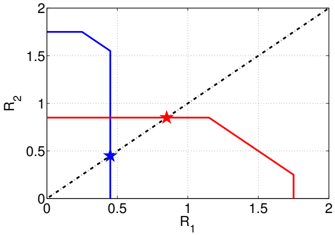

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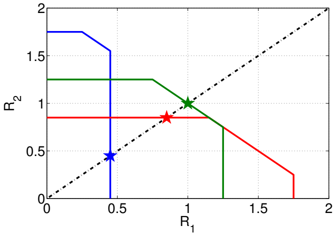

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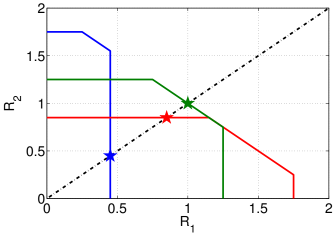

Well, there are three faces so... 1/3?

Symmetric vs. Sum Capacity

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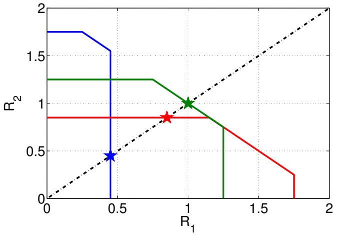
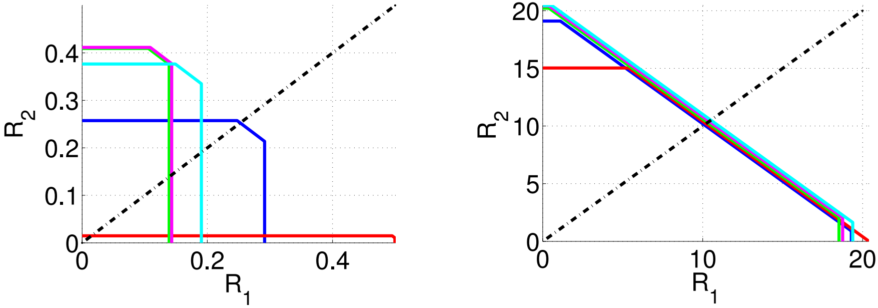


Correct answer, explanation can be improved...

What is Known

- $C_{\Sigma-\mathrm{sym}} \leq C_{\mathrm{sum}}$
- (Implicitly from the MAC-DMT): SNR $\rightarrow \infty \;\; \Rightarrow \;\; C_{\Sigma-\mathrm{sym}} \overset{w.h.p.}{\longrightarrow} C_{\mathrm{sum}}$



- Our goal: analyze the (finite SNR) distribution of $C_{\Sigma-\mathrm{sym}}$ given C_{sum}

Goal & Applications

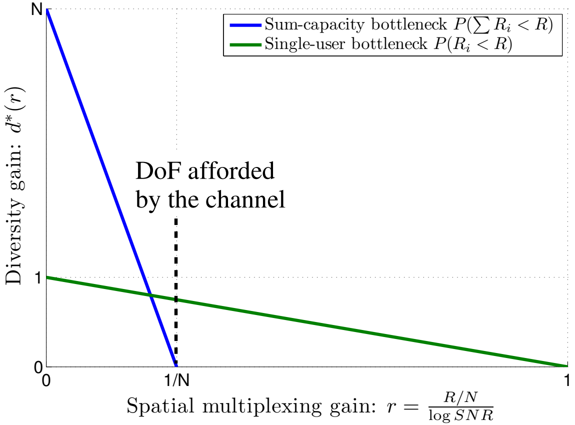
Q2: What is $\Pr\left(C_{\Sigma-\mathrm{sym}} < R \middle| C_{\mathrm{sum}} = c_{\mathrm{sum}}\right)$ **?**

Applications

- Simple MAC transmission protocol
  - Receiver learns channel gains of active users
  - Calculates Symmetric capacity and notifies transmitters to each transmit at rate R/N where $R < C_{\Sigma-\mathrm{sym}}$:
    - Trivial rate allocation
    - Minimal feedback
- Rayleigh open-loop outage probability
  - N active users
  - All users (when they are active) transmit at a common target rate R_t
  - Outage probability is then given by $\mathbb{E}_{C_{\mathrm{sum}}}[\Pr(C_{\Sigma-\mathrm{sym}} < NR_t | C_{\mathrm{sum}})]$

Let's recall the DMT of the MAC

Shouldn't be too bad...



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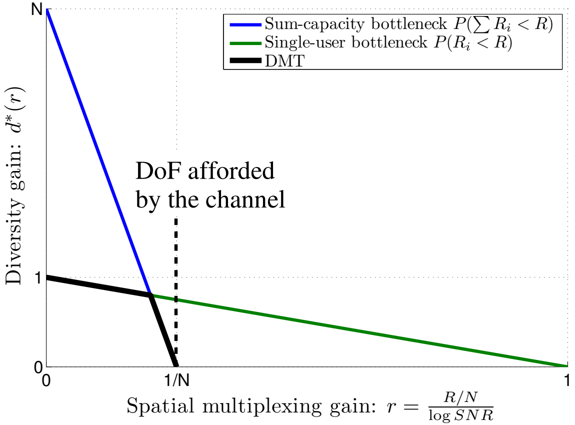
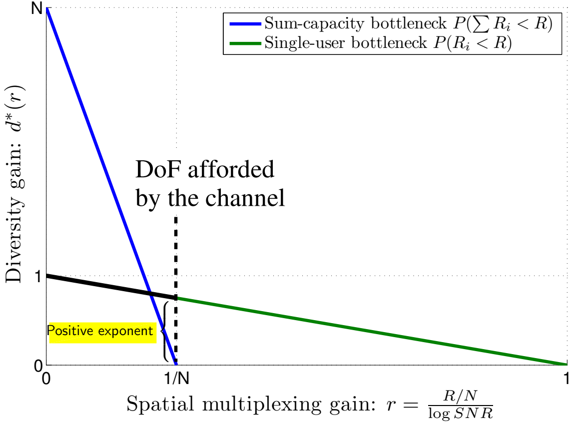


But, in our analysis/protocol we know C_{sum}...



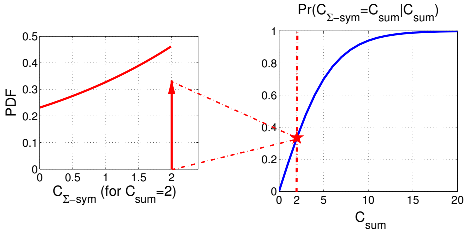
Bottom Line - Two-User Rayleigh MAC

Theorem 1

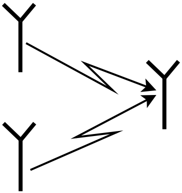
For a 1×2 Rayleigh MAC with sum capacity C_{sum}:

$$\Pr(C_{\Sigma-\text{sym}} < R | C_{\text{sum}}) = 2 \cdot \frac{2^{R/2} - 1}{2^{C_{\text{sum}}} - 1}; \; 0 \leq R \leq \boldsymbol{C_{\text{sum}}}$$

$$\Pr(C_{\Sigma-\text{sym}} = C_{\text{sum}} | C_{\text{sum}}) = 1 - \Pr(C_{\Sigma-\text{sym}} < C_{\text{sum}} | C_{\text{sum}})$$
$$= 1 - 2 \cdot \frac{2^{C_{\text{sum}}/2} - 1}{2^{C_{\text{sum}}} - 1}$$



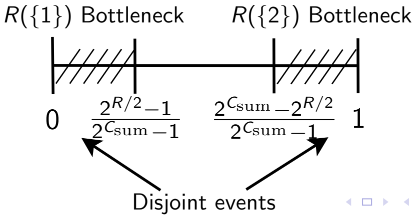
Sketch of Proof: Two-User Rayleigh MAC



- $h_i \sim \mathcal{CN}(0, \mathrm{SNR})$ and i.i.d $\Rightarrow \; |h_i|^2 \sim exp(\mathrm{SNR})$
- Normalize: $u_i = \frac{1}{\sqrt{2^{C_{\mathrm{sum}}}-1}} h_i$
- **Given** C_{sum}
    - $|u_1|^2 + |u_2|^2 = 1 \Rightarrow$ zero-sum game
    - $|u_i|^2$ given $|u_1|^2 + |u_2|^2 = 1$ is uniformly distributed over $[0, 1]$ (conditioning property of Poisson process)

Sketch of Proof: Two-User Rayleigh MAC

- $C_{\Sigma-\mathrm{sym}} = N \min_{S \subseteq \{1,\dots,N\}} R(\{S\}) = N \min_{S \subseteq \{1,\dots,N\}} \frac{1}{|S|} \log\left(1 + \sum_{i \in S} |h_i|^2\right)$
- Two users, given C_{sum}: $C_{\Sigma-\mathrm{sym}} = \min\left(2R(\{1\}), 2R(\{2\}), \xcancel{C_{\mathrm{sum}}}\right)$
- $R(\{i\}) = \log\left(1 + |u_i|^2 (2^{C_{\mathrm{sum}}} - 1)\right)$
- $\Pr\left(C_{\Sigma-\mathrm{sym}} < R | C_{sum}\right) =$
  $\Pr\left(|u_1|^2 < \frac{2^{R/2}-1}{2^{C_{\mathrm{sum}}}-1}\right) + \Pr\left(|u_1|^2 > \frac{2^{C_{\mathrm{sum}}}-2^{R/2}}{2^{C_{\mathrm{sum}}}-1}\right)$
- $\Rightarrow \Pr\left(C_{\Sigma-\mathrm{sym}} < R | C_{sum}\right) = 2\frac{2^{R/2}-1}{2^{C_{\mathrm{sum}}}-1}$



General N: The Bottleneck

- When $N > 2$:
  - There are more possible bottlenecks to check (but remember the DMT moral...)
  - Need to analyze

$$\Pr\left(R(\{S\}) < R | C_{\text{sum}}\right) =$$
$$\Pr\left(\frac{|S|}{N}\log\left(1 + (2^{C_{\text{sum}}} - 1)\sum_{i\in S}|u_i|^2\right) < R \,\Big|\, \sum |u_i|^2 = 1\right)$$

  - Possible bottlenecks $\{S\}$ are no longer disjoint
- Tool for analysis
  - Given C_{sum}, u_i can be viewed as elements from a row taken from a unitary matrix drawn from the CUE (Haar measure)
  - Edelman 05' - Singular value distribution of a truncated unitary matrix (eigenvalues have Jacobi/MANOVA distribution)
- $\Rightarrow$ lower and upper bounds

General N: The Bottleneck

Theorem 2 - distribution of a specific set

For a $1 \times N$ Rayleigh MAC with sum capacity C_{sum}, the outage probability for a set $S \subseteq \{1, 2, \ldots, N\}$ is

$$\Pr\left(R(\{S\}) < R | C_{\mathrm{sum}}\right) =$$

$$\Pr\left(\frac{|S|}{N}\log\left(1 + (2^{C_{\mathrm{sum}}} - 1)\sum_{i \in S}|u_i|^2\right) < R \;\middle|\; \sum |u_i|^2 = 1\right) =$$

$$\frac{\mathcal{B}\left(\frac{2^{R|S|/N}-1}{2^{C_{\mathrm{sum}}}-1}; |S|, N - |S|\right)}{\mathcal{B}(1; |S|, N - |S|)}$$

where $0 \leq R \leq C_{\mathrm{sum}}$ and $\mathcal{B}(x; a, b) = \int_0^x u^{a-1}(1-u)^{b-1}du$ is the incomplete beta function.

General N: The Bottleneck

- $\Pr\left(C_{\Sigma-\text{sym}} < R | C_{\text{sum}}\right) = \Pr\left(\min_{S \subseteq \{1,2,\ldots,N\}} R(\{S\}) < R | C_{\text{sum}}\right)$
- All sets with the same cardinality have the same outage probability
- $P_{\text{out}}(k, R) \triangleq \Pr(R(\{|S| = k\} < R | C_{\text{sum}})$
- Union bound can be used to bound overall probability

Theorem 3 - lower and upper bound for N Rayleigh MAC

For a $1 \times N$ Rayleigh MAC with sum capacity C_{sum}, the outage probability can be bounded as

$$\max_k P_{\text{out}}(k, R) \;\leq\; \Pr\left(C_{\Sigma-\text{sym}} < R | C_{\text{sum}}\right) \;\leq\; \sum_{k=1}^{N} \binom{N}{k} P_{\text{out}}(k, R)$$

Upper and Lower Bounds

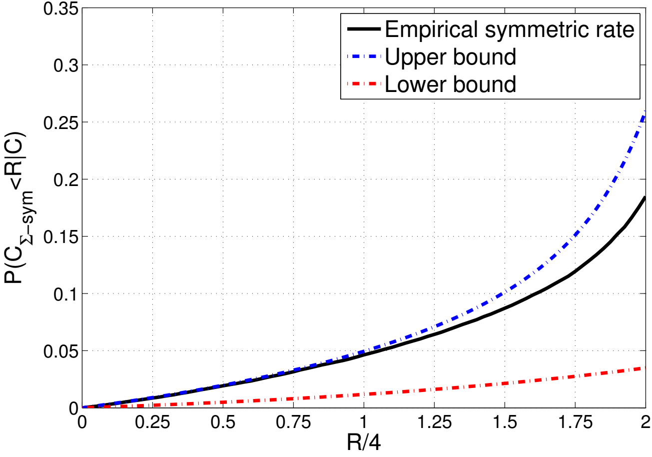


Figure: Bounds vs. Empirical error probability for 1×4 channel with $C_{\mathrm{sum}}/4 = 2$

Practical Scheme (NOMA): Two-User Example

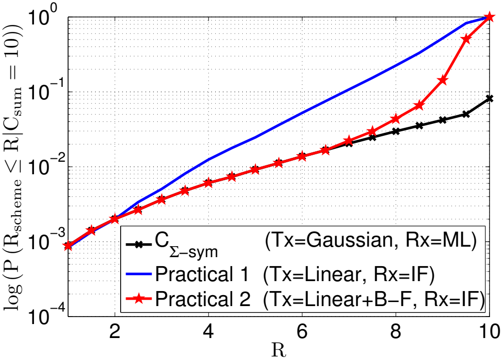


Figure: ML vs. IF for a two-user i.i.d. Rayleigh fading MAC with $C_{\text{sum}} = 10$.

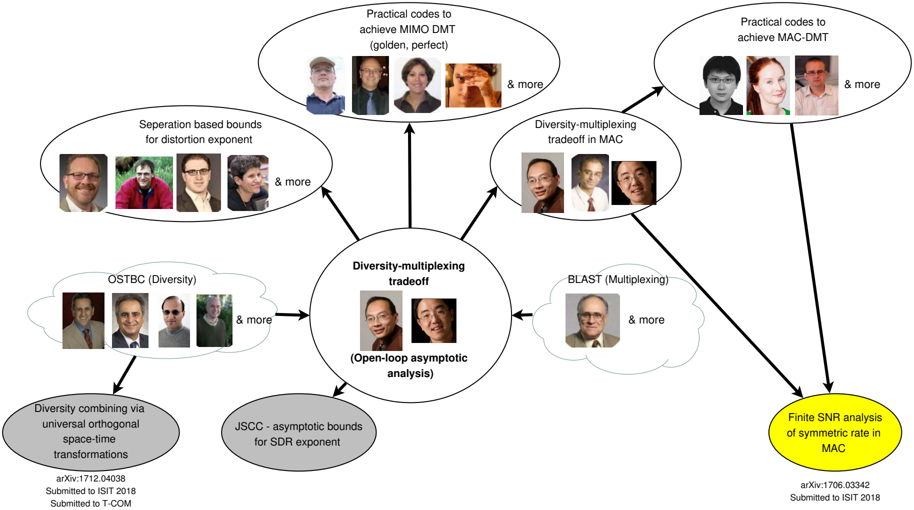
Ingredients

Integer-forcing (Rx)

- Beyond this talk
- All users use the same linear code
- Not sufficient

MAC-DMT (Tx)

- Hollanti, et. al., '11 (uncoded, asymptotic)
- Same linear code ⇒ need to use (different) "space"-time modulation
- Badr, et. al. '08



Slides available in https://domanovi.github.io/